Option Pricing on Cryptocurrency Underlying using Diffusion and Jump-Diffusion Stochastic Volatility Models

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Abstract

This paper aims at understanding the dynamics of pricing option on Cryptocurrencies - BTC and ETH using diffusion (Heston 1993) and jump diffusion models (Bates 1996). Cryptocurrency markets tend to show high volatility and abnormally high returns in a rally which begs two critical question, (1) How appropriate are current models in circulation in capturing dynamics of asset prices and their consequences on option prices and (2) What can market observed option data tell us about the investors in this de-centralised market places. I've tried to answer these two questions in a meaningful manner with demonstrable evidence from the three models in discussion, which can be treated as an incremental refinements over years in modelling asset prices and understanding option prices as quoted in market.

Keywords: Options, Cryptocurrency, SVCJ, Heston, Bates, Jump Diffusion Model, BTC, ETH

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1 Introduction

Decentralised digital currencies and peer-to-peer payment mechanisms have gained traction exponentially in recent years. Underlying technologies supporting such currencies have made them a likely candidate for **currency of the future** in some ways, shapes or forms. As the supporting infrastructure around digital currencies evolves and their adoption gains pace, we are seeing an increasing interest in derivatives market based on these digital currencies.

With each transaction in crypto-market essentially being a record on a de-centralised ledger, it has facilitated any user with a crypto wallet to become a participant in the derivatives market. This is materially different to what we see in traditional asset classes and traditional financial markets where a retail investor has to have a broker account and depending on their sophistication within retail segment a particular investor is permitted to trade using certain high risk instruments which might have a non-linear relationship to the underlying asset and is therefore "risky" to understand and manage.

Many exchanges as of the year 2022 permit retail as well as institutional investors to carry out trades in a common market place without an intermediary which has been the point of motivation towards writing this article. As seen in traditional financial markets where "smart money" seems to always have an upper hand, I'm keen to understand what are the implication on behaviour of option prices and volatilities where we've a truly open and de-centralised market place.

To better understand these implications, I've tried to compare equity and crypto option markets, using a set of three models and drawn some meaningful economic conclusions. The three models are chosen such that I'm able to modulate the underlying stochastic processes for asset pricing using a combination of diffusion, jump-diffusion and correlated jump-diffusion processes.

1.1 Cryptocurrency Market

Market cap of cryptocurrencies have steadily risen since inception, with local peaks noticeable around the end of pandemic years and understandably so, as seen below

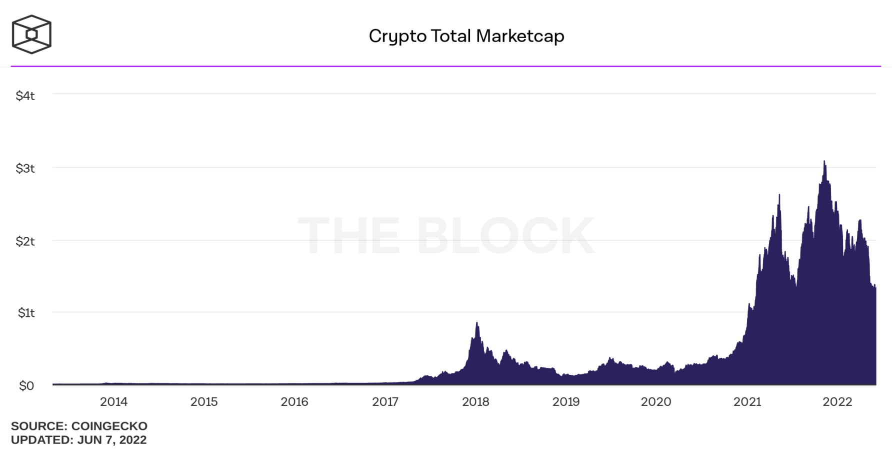

Fig. 1: Total Cryptocurrency Market Capitalisation

Looking particularly at the "household" cryptocurrencies i.e. BTC and ETH, since 2017, market has seen a steady upwards trend in unit price. This is not a surprise in hindsight with the amount of attention and coverage these currencies have enjoyed over recent number of years, albit, beginning of 2022 has seen significant drops in both currencies setting new records.

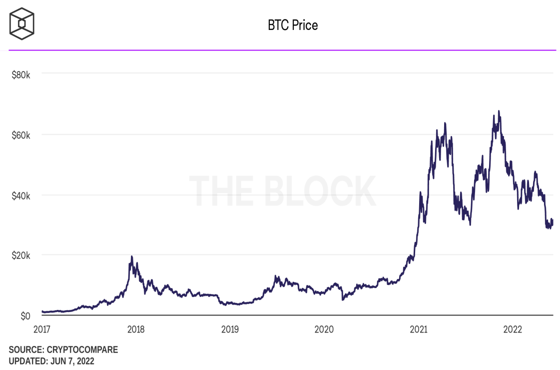

Fig. 2: BTC Price History

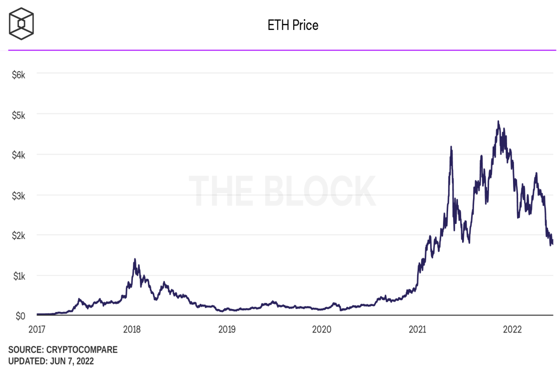

Fig. 3: ETH Price History

1.2 Market Structure

Write about miners, POS, POW how ETH is moving to other etc.etc.

2 BTC & ETH Options Market

Derivatives market in Cryptocurrencies has gained momentum with many regulated and unregulated exchanges in operations as of 2022. These exchanges offer access to retail and institutional investors with or without a crypto wallet, which has encouraged many early adaptors, speculators, techno-nerds and investors from all walks of life, thereby creating a much more healthier and heterogeneous mix of market participants.

2.1 Options

For the purposes of this paper, I'm focusing on options quoted on BTC and ETH futures which are traded on Deribit exchange. Deribit is an unregulated exchange with approximately 90% of traded volumes across BTC and ETH options.

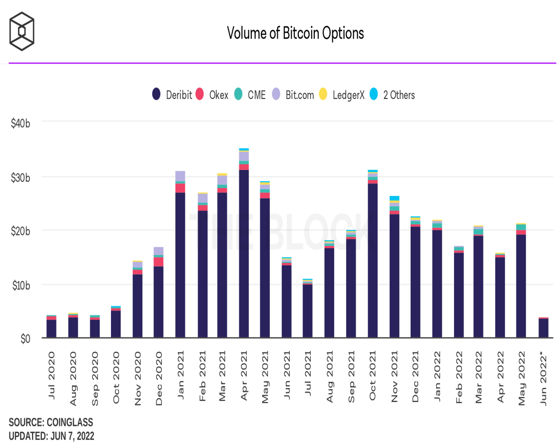

Fig. 4: BTC Options Volume

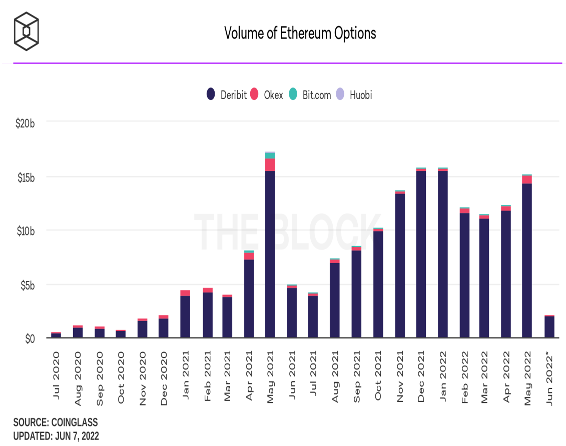

Fig. 5: ETH Options Volume

The option contract is written on a cash settled future contract on BTC and ETH, where maturity of option T_i^o matches that of underlying future contract T_i^f. Should any slice of maturity is missing in underlying futures contract strip, Deribit floats an option on a synthetic futures contract. Maturities for underlying futures and consequently for options range from one day to one year with unequally spread intervals. For each maturity slice, a number of strikes K_i^t are traded, creating a sufficiently dense grid. Therefore, in this article, I'll be analysing options quoted on futures and not directly on BTC/USD oe ETH/USD.

All options on Deribit are European style, which means they can only be exercised at expiry, unlike American style options, that can be exercised any time until expiry. Options on Deribit are also cash settled, which means when they are exercised it is only the profits that are paid.

The above construct present it's own set of challenges from option pricing perspective. Since the options are quoted in units of underlying futures contract such that the price in "base currency", say USD, is $x_f * F_t^T$, where x_f refers to units and F_t^T is the price of the underlying future contract at t, maturing at T, in USD.

3 Pricing and Model Dynamics

3.1 Stochastic Volatility: Heston Model

Heston model as published by Steven L. Heston [1] in 1993, assumes the spot price for underlying asset at any given time t follows a continuous diffusion process as described in equation (**??**), where $dW_t^{(S)}$ is a Weiner process

$$dS_t = \mu S_t dt + \sqrt{V}_t S_t dW_t^{(S)} \tag{3.1.1}$$

Heston further builds out that the volatility process which feeds the asset process follows an Ornstein–Uhlenbeck process, which results in a familiar CIR [2] (Cox, Ingersoll and Ross) process as described in (3.1.2) below,

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V}_t dW_t^{(V)} \tag{3.1.2}$$

The model further assumes the correlation between the two Weiner processes $dW_t^{(S)}$ and $dW_t^{(V)}$ is defined as ρ,

$$Cov(dW_t^{(S)}, dW_t^{(V)}) = \rho dt \tag{3.1.3}$$

3.1.1 Parameters

Some initial and sensible conditions are assumed on process S_t and V_t as defined below,

$$S(0) = S_0 > 0; V(0) = V_0 > 0 \tag{3.1.4}$$

Here S_t represents the price process for underlying at any time t and V_t is the instantaneous variance at any time t.
θ is the long run mean variance and κ corresponds to the speed of adjustment of volatility of volatility σ_v to the long run mean variance θ. The two Weiner processes $dW_t^{(S)}$ and $dW_t^{(V)}$ is defined under risk neutral measure $\tilde{\mathbb{P}}$ with instantaneous correlation ρ. The rate of return under risk neutral measure for underlying asset price process S_t is defined as μ.

3.1.2 Closed Form Solution: European Call

Under Heston [1] dynamics, any contingent claim, say X, will be a function of price of the underlying asset S, the volatility V and time t, which I'll express as $X_{(S,V,t)}$. This claim is assumed to be an European call for the purposes of this paper and therefore is only depend on the information known to the processes at or before time t, in other words, $X_{(S,V,t)}$ is adopted to filtration until t, where the value of the claim is dependent on price process at t denoted at S_t, the corresponding volatility process at t denoted at V_t and time t itself. It is not dependent on outcomes of each of the stochastic processes prior to t however those are known.

Therefore the Heston dynamics for European call will follow the PDE,

$$\frac{1}{2}VS^2\frac{\partial^2 X}{\partial S^2} + \rho\sigma_v VS\frac{\partial^2 X}{\partial S \partial V} + \frac{1}{2}\sigma_v^2 V\frac{\partial^2 X}{\partial V^2} + rS\frac{\partial X}{\partial S} + (\kappa(\theta - V) - \lambda(S,V,t))\frac{\partial X}{\partial V} - rX + \frac{\partial X}{\partial t} = 0 \tag{3.1.5}$$

As Heston pointed out in [1], we can let the market price of volatility risk, given by $\lambda(S, V, t)$, be equal to λV.

To construct an European Call with strike K and maturity T, the PDE for claim X (3.1.5) must follow the option boundary conditions as stated below [7],

$$X_{(S,V,t)} = max(S_t - K, 0) \tag{3.1.6}$$

$$X_{(S,V,t)} = 0 \quad \forall\, S = 0 \tag{3.1.7}$$

$$X_{(S,V,t)} = 1 \quad \lim_{S \to \inf} \tag{3.1.8}$$

$$rS\frac{\partial X}{\partial S}(S,V,t) + \kappa\theta\frac{\partial X}{\partial V}(S,V,t) - rX(S,V,t) + X_t(S,V,t) = 0 \quad \forall\, V = 0 \tag{3.1.9}$$

$$X_{(S,V,t)} = S \quad \lim_{V \to \inf} \tag{3.1.10}$$

Value of an European call can now be written as

$$c_{(S,V,t)} = e^{-r\tau}E[(S_t - K), 0] = e^x P_1(x,V,t) - e^{-r\tau}KP_2(x,V,t) \tag{3.1.11}$$

where $x = lnS$ and $\tau = T - t$

3.1.3 Characteristic Function

Heston [1] famously guessed the functional form of the characteristic function

$$f(S,V,t;\phi) = e^{C(T-t;\phi)+D(T-t;\phi)V+i\phi S} \tag{3.1.12}$$

The coefficients C and D are of the form,

$$C(\tau;\phi) = r\phi i\tau + \frac{\alpha}{\sigma^2}\left((b_j - \rho\sigma\phi i + d)\tau - 2ln\left[\frac{a - ge^{d\tau}}{1-g}\right]\right) \tag{3.1.13}$$

$$D(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right] \tag{3.1.14}$$

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \tag{3.1.15}$$

$$d = \sqrt{(\rho\sigma\phi\tau - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)} \quad \forall\, j = 1,2 \tag{3.1.16}$$

The probability can then be computed by integrating the real part,

$$P(S,V,T;ln[K]) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi ln[k]}f_j(S,V,T;\phi)}{i\phi}\right] \tag{3.1.17}$$

3.2 Stochastic Volatility with Jumps: Bates Model

Bates [3] in 1996 extended the Heston model dynamics to include for jumps in asset prices, which would account for sudden moves in underlying. When analysing asset processes in context of cryptocurrencies, where historically markets have known to be highly volatile and significant jumps in levels of various mainstream coins are well documented (i.e. BTC, DOGE), an appropriate adjustment factor on asset price process seems like a logical approach to better model the cryptocurrency dynamics.

Bates describes the asset price process as follows,

$$dS_t = (r - q - \lambda\mu_J)S_t dt + \sqrt{V}_t S_t dW_t^{(S)} + J_t S_t dN_t \tag{3.2.1}$$

where r is the continuous risk free rate of return on asset S and q is the continuous dividend yield.

As of year 2022, a "dividend" phenomenon is not directly observed on traditional buy-and-hold type of trades on cryptocurrencies, therefore for the purposes of this paper, I'll be setting the value for $q = 0$. Whilst there are some exchanges and DAOs (Decentralised Autonomous Organisations) that allow investors to "stake" their coins and earn a APY (Annualised Percentage Yield), which could be seen as a viable proxy for dividends, it is important to note that these staking rewards are not available to investors who simply buy-and-hold cryptocurrencies in their hot or cold wallets from exchanges nor are these rewards distributed to option holders/writers. More-over, exchanges and DAO pay out these rewards in their native token and which are assumed to be independent assets to the one of which I'm assessing the option price behaviour for.

As with Heston model, Bates model dynamics assumes a CIR [2] fashioned volatility process, the parameters of which are explained in section 3.1

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V}_t dW_t^{(V)} \tag{3.2.2}$$

$$Cov(dW_t^{(S)}, dW_t^{(V)}) = \rho dt \tag{3.2.3}$$

The random process exciting jumps N_t in asset prices is assumed to follow a poisson with intensity λ, hence the probability of jump size equal to 1 is λdt. In (3.2.1), J is the random jump size of which is expressed as percentage and it's realisation is conditioned on its occurrence. Similar to Merton's jump model [4], the logarithm of jump size J_t is distributed as Gaussian as described in (3.2.5)

$$P(dN = 1) = \lambda dt \tag{3.2.4}$$

$$log(1 = J_t) = \mathcal{N}(log(1 + \mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2) \tag{3.2.5}$$

3.2.1 Parameters

4 Model Calibration and Results

Implementing any model to price options faces the challenge to reproduce the observed market prices $C^m_{K,T}$ for a set of strikes and maturities. The parameters used in model to control the behaviour of various factors / moments then need to be estimated in such a manner that the model overall reproduces the market observed prices. This process of determining the model parameters is often known as model calibration.

Whilst there are number of methods and techniques that can be employed to determine the model parameters, the estimation method of such parameters is extremely critical as this determines the overall stability of the model, should the underlying data or observed market prices change materially in events of shocks and crises. The major challenge in calibrating any model is the fact that model parameters aren't directly market observables and therefore the input data available is incomplete it in observed form to identify required parameter set.

In this paper, and as often observed as a common market practice, I'll employ several various minimisation techniques to estimate the model parameters for calibration. The idea is to re-generate market observed prices $C^m_{K,T}$ using an objective function (the closed form solution of European calls) and minimising the squared difference between the two, thereby spitting out the optimal set of model parameters that justify the market observed price, or a price as close to market observed price as possible.

$$\text{minimize } f(x)$$
$$f(x) = \sum_{i=1}^{N} w_i |C^m_{K,T} - c_{S,V,K,T}|^2$$

where, $c_{(S,V,K,T)}$ is the model price for European call with underlying asset price process S, volatility process V, strike K and maturity T. The minimisation of error (differences between prices squared) is computed over N options that are used with weight of each being w_i.

Depending on the model, the minimisation techniques may vary, which I've described in the individual model calibration subsection throughout this section.

4.1 Heston Model

4.1.1 Minimisation Routine

4.2 Bates Model

4.2.1 Minimisation Routine

5 Conclusions

A SVCJ

A.1 Stochastic Volatility with Correlated Jumps: SVCJ

As suggested by Hou et al. [5], a continuous-time model suggested by Duffie et al. [6] seems to be the most appropriate model candidate. This continuous-time model accounts for correlated jumps between a stochastic volatility process and a return process thereby adding one additional parameter to the mix. Such a stochastic volatility correlated jump model (SVCJ) has one additional parameter ρ_j, which is an extension to the Bates [3] which models a single jump in underlying asset process.

$$dlogS_t = \mu dt + \sqrt{V}_t dW_t^{(S)} + J_t^y dN_t \tag{A.1.1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V}_t dW_t^{(V)} + J_t^v dN_t \tag{A.1.2}$$

$$Cov(dW_t^{(S)}, dW_t^{(V)}) = \rho dt \tag{A.1.3}$$

$$P(dN = 1) = \lambda dt \tag{A.1.4}$$

B Metropolis-Hastings Algorithm

Sampling from a non-standard distribution with probability density $P(x)$ can be challenging when performing a Markov Chain Monte Carlo. The Metropolis-Hastings algorithm provides an elegant way to sample a non-standard distribution provided we know a function $f(x)$ proportional to the density $P(x)$ and values of $f(x)$ can be calculated.

The algorithm works with a fundamental premise that more and more samples values are produced, then the distribution of produced sample closely resembles the required non-standard distribution with density $P(x)$.

The samples are drawn iteratively in such that the next sample value is based on the current sample value, thus forming a Markov Chain. A certain number of initial draws are "burnt" to ensure that the process starts with sufficient samples drawn so as to resemble the distribution to desired distribution as close as possible. With the next sample, a certain probability of acceptance is attached, which is determined by comparing the function $f(x)$ value at current and next proposed candidate sample. If the proposed candidate value is discarded, then the current value is used to repropose a new candidate in an iterative fashion.

An end state is said to be achieved in this processes, when the Markov process asymptotically reaches a unique stationary distribution $\pi(x)$ such that, $\pi(x) = P(x)$.

Let us define a Markov process with transition probability of $P(x'|x)$, which is defined as transition probability of going from $x \to x'$. Now, a unique stationary distribution $\pi(x)$ must exist, which demands 2 conditions to be met.

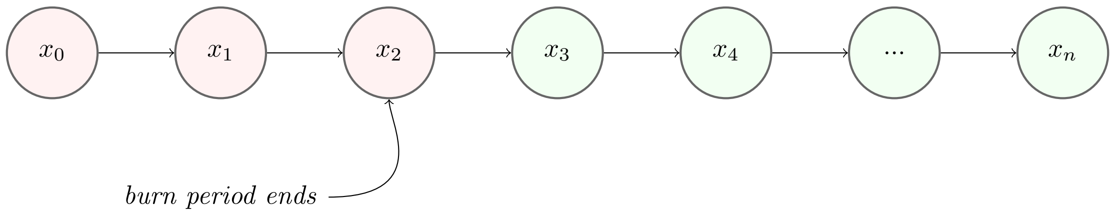

Fig. 6: Sampling from distribution with unknown probability density $P_{(x)}$

- **Detailed Balance:** A weak condition that requires each transition is reversible, i.e. $x \to x'$ and $x' \to x$ are permissible transitions for a given pair of samples x, x'. And therefore, the transition probability from either state to another must be equal, i.e. $\pi(x)P(x'|x) = \pi(x')P(x|x')$.
- **Uniqueness:** A condition requiring every state (x) must be aperiodic, i.e. process not returning to same state (x) after a fixed set of iterations and positive recurrent, i.e. expected number of iterations to return to the same state are finite. This ensures that the state is ergodic in nature.

The detailed balance can be expressed as

$$\frac{P(x'|x)}{P(x|x')} = \frac{P(x')}{P(x)} \tag{B.0.1}$$

How-ever the transition to next state is a two step step process: (1) Proposing a sample candidate (x') and (2) there-after accepting or rejecting the sample candidate.

Let $Q(x'|x)$ be the conditional probability from the proposal distribution Q, for proposing a state change $x' \to x$, and let $A(x'|x)$ be the probability of acceptance of newly proposed state (x').

$$P(x'|x) = Q(x'|x)A(x'|x) \tag{B.0.2}$$

Using equation (B.0.1),

$$\frac{A(x'|x)}{A(x|x')} = \frac{P(x')Q(x|x')}{P(x)Q(x'|x)} \tag{B.0.3}$$

An acceptance ratio can be chosen such that above condition is met.

$$A(x'|x) = \min\left(1, \frac{P(x')Q(x|x')}{P(x)Q(x'|x)}\right) \tag{B.0.4}$$

Therefore, either $A(x'|x) = 1$ or $A(x|x') = 1$.

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